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# Finite-size corrections in the $XY$ model with a uniform magnetic field and a boundary field

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**Abstract.** The one-dimensional $XY$ model with a uniform magnetic field and a boundary field is introduced. The present model is solved analytically at its critical point. Using the present analytical result, the finite-size corrections of the ground-state energy and the low-lying excited energies are evaluated. The partition function of the present model is also evaluated in the scaling limit. Through this calculation, the central charge and the conformal weights of the primary fields in the present model are obtained analytically.

## 1. Introduction

The investigation of the one-dimensional $XY$ model with a uniform magnetic field has a history dating back half a century. The present model is described by the following Hamiltonian under the periodic boundary condition:

$$\mathcal{H} = -\sum_{j=1}^{L}\left(J_x\sigma_j^x\sigma_{j+1}^x + J_y\sigma_j^y\sigma_{j+1}^y + \Gamma\sigma_j^z\right) \tag{1.1}$$

where the symbol $\sigma_j^a$ denotes the $a$-component of the Pauli matrices at site $j$. In particular, the present model corresponds to the transverse Ising model when $J_y = 0$, and this model also describes the $XY$ model when $\Gamma = 0$.

In the present paper, we discuss critical phenomena of the one-dimensional $XY$ model with a uniform magnetic field and a boundary field. In particular, we focus on the critical phenomena in the following two special cases, namely the transverse Ising limit and the ordinary $XY$ case. As is well known, these critical phenomena belong to different universality classes. (This fact is briefly explained later.) Namely, the present model shows two kinds of critical phenomena in different regions of the relevant parameters. What influence does the boundary field exert on these critical phenomena of different universality classes? The physical reason for studying the present model is to answer this question.

Although we do not trace the whole history here, we briefly review several properties of the present model, which are related to the present investigation, as preliminaries of our discussion. As was pointed out by Nambu [1], the present model can be expressed in terms of spinless fermions. In fact, the $XY$ model [3, 4] and the transverse Ising model [4, 5] were solved by means of the Jordan-Wigner transformation [2] to obtain the dispersion relation. Then, it was found that the critical points of the ground-state phase transitions are given by $J_x = J_y$ and $J_x = \Gamma$ for the $XY$ model and the transverse Ising model, respectively. Moreover, one of the present authors (MS) generalized the present model and exactly solved the generalized model in terms of fermions [6]. Here we remark that the transverse Ising model and the $XY$ model can be solved not only under the periodic boundary condition but also under the open boundary condition [3,5]. It was proved by MS [6,7] that the ground state of the present model (1.1) is equivalent to Onsager's Ising model in a region of the parameters $J_x$, $J_y$ and $\Gamma$. (It was also proved [8] that the present model is equivalent to the eight-vertex model with the free-fermion condition (not Baxter's eight-vertex model) in a parameter region.) Using the relationship obtained by MS [6,7], we find that the transverse Ising model with $J_x = \Gamma$ corresponds to the two-dimensional Ising model at its phase-transition point, but that the $XY$ model with $J_x = J_y$ does not. Therefore, we can identify the critical transverse Ising model with the critical model of the central charge $c = \frac{1}{2}$, because the two-dimensional Ising model at its critical point corresponds to such a minimal model in the conformal field theory (CFT) [9]. We can also evaluate the central charge by taking a continuum limit. In this limit, the critical transverse Ising model is transformed into the massless Majorana fermion field which has $c = \frac{1}{2}$ (see [10], for example). In the continuum limit, we can transform the isotropic $XY$ model into the massless Dirac fermion field which is equivalent to a massless scalar field [11,12] and has $c = 1$ (see [10], for example). This means that the isotropic $XY$ model belongs to the universality class of the two-dimensional plane-rotor model.

In the present paper, we discuss the critical phenomena of the one-dimensional $XY$ model with a uniform magnetic field and a boundary field, using the finite-size scaling technique [13–15] based on the boundary CFT [16]. The present model under the periodic boundary condition and under the open boundary condition has been discussed in the context of finite-size scaling based on the CFT or the boundary CFT. See [17–22], for example.

In section 2, we introduce the one-dimensional $XY$ model with a uniform magnetic field and a boundary field and solve it analytically. Using the analytical solution, we evaluate the finite-size corrections for the ground-state energy of the present model in section 3. We also obtain the low-lying effective Hamiltonian of the model. We evaluate the central charge and the surface exponent of the present model without the boundary field, in section 4. We calculate the partition function of the present model, which is described by the characters of the primary fields of the Virasoro algebra, in section 5. By using the partition function thus obtained, we discuss the operator contents of the present model, namely what conformal dimensions the primary fields have, in sections 6, 7 and 8. Through the discussion, we can evaluate how the conformal dimensions of the primary fields, which exist in the present model, depend on the boundary fields. In section 9, taking a continuum limit of the present lattice model, we obtain the classical two-dimensional system on a half-plane to express the boundary field at the edge of the plane.

## 2. Analytical solution of the model

In the present section, we introduce the relevant model and solve it analytically.

The one-dimensional $XY$ model with a uniform magnetic field and a boundary field is described by the following Hamiltonian:

$$\mathcal{H} = -\sum_{j=1}^{L-1} \left\{ \tfrac{1}{2}(1+\alpha)\sigma_j^x \sigma_{j+1}^x + \tfrac{1}{2}(1-\alpha)\sigma_j^y \sigma_{j+1}^y \right\} - \gamma \sum_{j=1}^{L} \sigma_j^z - h\sigma_1^z - h'\sigma_L^z. \tag{2.1}$$

Here, the parameters $h$ and $h'$ denote the magnitudes of the boundary fields. In the present paper, we discuss the present model on a linear chain with $L$ sites, where $L$ is a sufficiently large even number.

By using the Jordan–Wigner transformation [2], we transform the present model (2.1) into the following fermion model:

$$\mathcal{H} = -\sum_{j=1}^{L-1} \left\{ \left( c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j \right) + \alpha \left( c_j^\dagger c_{j+1}^\dagger + c_{j+1} c_j \right) \right\}$$

$$-2\gamma \sum_{J=1}^{L} c_j^\dagger c_j - 2h c_1^\dagger c_1 - 2h' c_L^\dagger c_L + L\gamma + 2h. \tag{2.2}$$

Here, the symbol $c_j^\dagger$ $(c_j)$ denotes the creation (annihilation) operator of a spinless fermion at site $j$.

We diagonalize the present fermion model with a boundary. According to Lieb *et al* [3], if we express the Hamiltonian (2.2) in the following form:

$$\mathcal{H} = \sum_{ij} \left\{ c_i^\dagger A_{ij} c_j + \tfrac{1}{2} \left( c_i^\dagger B_{ij} c_j^\dagger + \text{h.c.} \right) \right\} + E_0 \tag{2.3}$$

it can be rewritten by a canonical transformation

$$\mathcal{H} = \sum_k \omega_k \left( \xi_k^\dagger \xi_k - \tfrac{1}{2} \right) + \frac{1}{2} \sum_i A_{ii} + E_0. \tag{2.4}$$

Here, $\{\omega_k^2\}$ are the eigenvalues of the matrix $M \equiv (A-B)(A+B)$. Thus, instead of diagonalizing the $2^L$-dimensional Hilbert space of the relevant Hamiltonian, we have only to diagonalize the $L$-dimensional matrix $M$. The symbol $\xi_k$ corresponds to the normal-mode fermion with the energy $\omega_k$. In the normal-mode expression (2.4), the quantity $\frac{1}{2}\sum_i A_{ii} + E_0$ of our model is equal to zero. An explicit form of the matrix $M$ is shown in Appendix A. For the $XY$ model $(\gamma = 0)$ under the free boundary condition, in other words, without the boundary field $(h = 0)$, a scheme for diagonalizing the matrix $M$ was shown by Lieb *et al* [3]. Following their scheme, Pfeuty [5] derived the eigenvalues of $M$ of the transverse Ising model $(\alpha = 1)$ under the free boundary condition $(h = h' = 0)$. Moreover, Boccara and Sarma [23] and Micnas and Kowalewski [24] solved the transverse Ising model with the boundary field. We remark that Bugrij and Shadura [20] and Bariev and Peschel [21] solved the transverse Ising model with an $x$-direction field or an equivalent model.

In the remaining part of the present paper, we discuss the following two critical models with the boundary field: (i) the critical transverse Ising model $(\alpha = \gamma = 1)$, and (ii) the isotropic $XY$ model $(\alpha = \gamma = 0)$. In these cases, we can extend the scheme [3] of Lieb *et al* to diagonalize the matrix $M$. Detailed calculations are shown in appendix A. The analytical results thus obtained are given as follows:

(i) In the critical transverse Ising model $(\alpha = \gamma = 1)$, the dispersion relation is given by

$$\omega_k = 4\cos\frac{k}{2} \tag{2.5}$$

where $k$ is one of the roots of the following equation [24]:

$$e^{(2L+1)ki} \left( \frac{1 - ((1+h)^2 - 1)e^{-ik}}{1 - ((1+h)^2 - 1)e^{+ik}} \right) \left( \frac{1 - ((1+h')^2 - 1)e^{-ik}}{1 - ((1+h')^2 - 1)e^{+ik}} \right) = 1. \tag{2.6}$$

(ii) In the isotropic $XY$ model $(\alpha = \gamma = 0)$, we have

$$\omega_k = 2\cos k. \tag{2.7}$$

Here $k$ is shown to satisfy one of the following equations:

$$\begin{cases} e^{(2L+2)k\,i}\left(\dfrac{1-2he^{-ik}}{1-2he^{+ik}}\right)\left(\dfrac{1-2h'e^{-ik}}{1-2h'e^{+ik}}\right)=1 \\ \text{or} \\ e^{(2L+2)ki}\left(\dfrac{1+2he^{-ik}}{1+2he^{+ik}}\right)\left(\dfrac{1+2h'e^{-ik}}{1+2h'e^{+ik}}\right)=1. \end{cases} \tag{2.8}$$

We solve these equations to obtain the allowed $\{k\}$ as the $L$ roots. Though the roots are not always real numbers, $\{\omega_k^2\}$ are always positive semi-definite. In the rest of the paper, we restrict our discussion to $h = h' \geqslant 0$.

## 3. Finite-size corrections in the models

In the present section, we evaluate the finite-size corrections for the ground-state energy in each model, by using the analytical solution obtained in the previous section. Moreover, we derive a low-lying effective Hamiltonian for each case.

As was shown in the previous section, each of the models takes the following form, after the normal-mode expansion,

$$\mathcal{H} = \sum_k \omega_k \left(\xi_k^\dagger \xi_k - \tfrac{1}{2}\right). \tag{3.1}$$

Therefore, the ground-state energy is described as

$$E_g = -\frac{1}{2}\sum_{m=1}^{L} \omega_{k_m} \tag{3.2}$$

where the symbol $k_m$ denotes the $m$th root of (2.6) or (2.8). We can evaluate the finite-size corrections of $E_g$ using the Euler–Maclaurin formula

$$\frac{1}{L}\sum_{m=m_1}^{m_2} f\left(\frac{m}{L}\right) = \int_{(m_1-\frac{1}{2})/L}^{(m_2+\frac{1}{2})/L} f(x)\,dx - \frac{1}{24L^2}\left(f'\left(\frac{m_2+\frac{1}{2}}{L}\right) - f'\left(\frac{m_1-\frac{1}{2}}{L}\right)\right) + o\left(\frac{1}{L^2}\right). \tag{3.3}$$

By expanding $E_g$ with respect to the powers of $L$, we obtain the following results:
(i) In the critical transverse Ising model $(\alpha = \gamma = 1, h \geqslant 0)$

$$E_g = eL + f - \frac{\pi v}{24L}\frac{1}{2} + o\left(\frac{1}{L}\right) \tag{3.4}$$

where

$$e = -\frac{4}{\pi} \qquad f = 1 - \frac{2}{\pi} - \frac{4}{\pi}F((1+h)^2 - 1). \tag{3.5}$$

(ii) In the isotropic $XY$ model $(\alpha = \gamma = 0, h \geqslant 0)$

$$E_g = eL + f - \frac{\pi v}{24L}\left(1 - 3\Theta^2(2h)\right) + o\left(\frac{1}{L}\right) \tag{3.6}$$

where

$$e = -\frac{2}{\pi} \qquad f = 1 - \frac{2}{\pi} - \frac{2}{\pi}F((2h)^2). \tag{3.7}$$

Here, the functions $F(x)$ and $\Theta(x)$ are defined as follows:

$$F(x) = -1 - \frac{x^2-1}{4x}\frac{\sqrt{4x}}{|1-x|}\tan^{-1}\left(\frac{\sqrt{4x}}{|1-x|}\right) \tag{3.8}$$

and

$$\Theta(x) = \begin{cases} \dfrac{4}{\pi}\arctan x & \text{for } x \leqslant 1 \\ \dfrac{4}{\pi}\arctan 1/x & \text{for } x > 1 \end{cases} \tag{3.9}$$

where $\tan^{-1}$ takes the principal branch arctan for $x \leqslant 1$ and takes the branch arctan $-\pi$ for $x > 1$.

In each model, we can recognize the sound velocity $v$ as 2 from the dispersion relation. When $L \to \infty$, the dispersion relation has zero at $k = \pi$ ($k = \pi/2$) for the critical transverse Ising model (the isotropic $XY$ model). The velocity is determined as a tangent of the dispersion relation about zero, namely $\omega_k \sim v|k-k_0|$ for $k \sim k_0$, where $k_0 = \pi$ ($k_0 = \pi/2$) for for the critical transverse Ising model (the isotropic $XY$ model).

Moreover, we can derive the low-lying effective Hamiltonian by expanding the dispersion relation about zero. In this region, we can linearize the dispersion relation as follows:

(i) for the critical transverse Ising model ($\alpha = \gamma = 1, h \geqslant 0$)

$$\mathcal{H}_{\text{eff}} = \frac{\pi v}{L}\left\{\sum_{m=0}^{\infty}\left(m+\tfrac{1}{2}\right)\xi_m^\dagger\xi_m\right\} - \frac{\pi v}{24L}\frac{1}{2} + o\left(\frac{1}{L}\right); \tag{3.10}$$

(ii) for the isotropic $XY$ model ($\alpha = \gamma = 0, h \geqslant 0$)

$$\begin{aligned}\mathcal{H}_{\text{eff}} = \frac{\pi v}{L}&\left\{\sum_{m=0}^{\infty}\left(m+\tfrac{1}{2}+\tfrac{1}{2}\Theta(2h)\right)\xi_m^\dagger\xi_m + \left(m+\tfrac{1}{2}-\tfrac{1}{2}\Theta(2h)\right)\eta_m^\dagger\eta_m\right\} \\ &-\frac{\pi v}{24L}\left(1-3\Theta^2(2h)\right) + o\left(\frac{1}{L}\right)\end{aligned} \tag{3.11}$$

where we drop the bulk energy $eL$ and the surface energy $f$ terms and neglect higher-order terms. In the latter model, we have two kind of fermions corresponding to two equations in (2.8).

In particular, the first excited gap of each model is shown to be

$$\Delta E_1 = \frac{\pi v}{L}\frac{1}{2} \tag{3.12}$$

$$\Delta E_1 = \frac{\pi v}{L}\frac{1}{2}\left(1-\Theta(2h)\right) \tag{3.13}$$

respectively.

In the critical transverse Ising model, the finite-size corrections for the energies do not depend on $h$. In contrast, the corresponding quantities of the isotropic $XY$ model change as functions of the boundary field.

## 4. Central charge and surface exponent for $h = 0$

In the present section, we discuss the critical phenomena of these systems by using the results obtained in section 3. As is well known [13], if the conformal dimension of the ground state is equal to zero, the finite-size corrections of the ground-state energy $E_g$ and the first excited gap above the ground state $\Delta E_1$ are expressed as follows:

$$E_g = eL + f - \frac{\pi v c}{24L} + o\left(\frac{1}{L}\right) \tag{4.1}$$

and

$$\Delta E_1 = \frac{\pi v x_s}{L} + o\left(\frac{1}{L}\right) \tag{4.2}$$

respectively. Here, $e$ and $f$ denote the ground-state energy per site and the surface energy per site, respectively. We denote the central charge and the surface exponent by the symbols $c$ and $x_s$, respectively.

For $h = 0$, we can recognize the central charge and the surface exponent from the above results, as follows:
(i) for the critical transverse Ising model ($\alpha = \gamma = 1, h = 0$)

$$c = \tfrac{1}{2} \qquad \text{and} \qquad x_s = \tfrac{1}{2}; \tag{4.3}$$

(ii) for the isotropic $XY$ model ($\alpha = \gamma = 0, h = 0$)

$$c = 1 \qquad \text{and} \qquad x_s = \tfrac{1}{2}. \tag{4.4}$$

In order to identify which operator acquires the surface exponent $x_s$, we consider the following correlation function as shown in [23], which denotes the imaginary-time correlation of the order-parameter operator at the boundary site:

$$\Gamma_1(\tau) \equiv \langle 0|\sigma_1^x(\tau)\sigma_1^x(0)|0\rangle = \sum_k (f_1(k))^2\, e^{-\tau\omega_k} \tag{4.5}$$

$$\sigma_j^x(\tau) = e^{\tau\mathcal{H}}\sigma_j^x e^{-\tau\mathcal{H}}. \tag{4.6}$$

Here the symbol $f_1(k)$ denotes the first component of the eigenvector which belongs to the eigenvalue $\omega_k^2$ of the matrix $M$. The ground state is described by the symbol $|0\rangle$. We can evaluate the asymptotic behaviour of the correlation function for the both models as follows:

$$\Gamma_1(\tau) \sim \int_{k_0}^{\infty} e^{-\tau|k-k_0|}\, dk \sim \frac{1}{\tau} \qquad \text{for } \tau \gg 1. \tag{4.7}$$

This means that the critical exponent $\eta_{\parallel}$ equals unity, namely $x_s = \eta_{\parallel}/2$ equals $\frac{1}{2}$ [25]. Thus, we can expect that the critical exponent of the present correlation function corresponds to the surface exponent obtained from the finite-size correction.

## 5. Partition functions of the models

In order to discuss the operator contents of each model, we evaluate the partition function, which is described by the characters of the Virasoro algebra [13].

We evaluate the following partition function as a function of $h$:

$$Z = \mathrm{Tr}\,\exp\left(-\frac{\hat{H}}{T}\right). \tag{5.1}$$

Here, the Hamiltonian $\hat{H}$ is defined as $\hat{H} = \mathcal{H} - eL - f$, using the parameter $q \equiv \exp(-\pi v/LT)$. As is well known [10], the present partition function is described as

$$Z = \sum_{\Delta} \mathcal{N}_{\Delta} \chi_{\Delta} \tag{5.2}$$

$$\chi_{\Delta} = q^{-\frac{c}{24}+\Delta} \sum_{N=0}^{\infty} d_{\Delta}(N) q^{N} \tag{5.3}$$

where the symbol $\chi_{\Delta}$ denotes the character of the highest-weight irreducible representation of the Virasoro algebra. The central charge and the conformal dimension of the primary field are described as $c$ and $\Delta$, respectively. We describe the degeneracy of the states at the $N$th level as $d_{\Delta}(N)$. The symbol $\mathcal{N}_{\Delta}$ denotes a non-negative integer. By evaluating the partition function, we can recognize the operator contents of the model, i.e. what primary field exists in the present model.

By using the low-lying effective Hamiltonian (3.10), (3.11) obtained in section 3, we obtain the following results in the scaling limit $q \sim 0$:

$$Z_{\text{transverse Ising}}(h) = q^{-\frac{1}{48}} \prod_{n=0}^{\infty} \left(1 + q^{n+\frac{1}{2}}\right) \tag{5.4}$$

and

$$Z_{XY}(h) = q^{-\frac{1}{24}+\frac{\Theta^2}{8}} \prod_{n=0}^{\infty} \left(1 + q^{n+\frac{1}{2}-\frac{\Theta}{2}}\right) \prod_{n=0}^{\infty} \left(1 + q^{n+\frac{1}{2}+\frac{\Theta}{2}}\right). \tag{5.5}$$

We remark that the latter partition function can be rewritten as follows, by using Jacobi's triple product identity:

$$Z_{XY}(h) = \frac{1}{\eta(q)} \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}(n+\frac{\Theta}{2})^2} \tag{5.6}$$

where

$$\eta(q) \equiv q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^{n}). \tag{5.7}$$

In this scaling limit, the partition function of the critical transverse Ising model does not depend on the boundary field $h$. In contrast, in the isotropic $XY$ model, the partition function depends on $h$ through the function $\Theta(2h)$.

## 6. Operator contents of the transverse Ising model

In the present section, we discuss the operator algebra of the critical transverse Ising model with a boundary field.

The partition function of the two-dimensional Ising model, whose boundary conditions are open in one direction and periodic in the other, is given as follows [20, 26]:

$$Z_{\text{2D Ising}} = q^{-\frac{1}{48}} \prod_{n=0}^{\infty} \left(1 + q^{n+\frac{1}{2}}\right) = \chi_0 + \chi_{1/2} \tag{6.1}$$

where $\chi_0$ and $\chi_{1/2}$ are two of the three characters of the Virasoro algebra with $c = \frac{1}{2}$. The other character is $\chi_{1/16}$, as is well known. In the scaling limit, we can find the following relationship:

$$Z_{\text{transverse Ising}}(h) = Z_{\text{2D Ising}} \qquad \text{for } \forall\, h \ (> 0). \tag{6.2}$$

We find that for any $h$, the critical transverse Ising model contains two kind of primary fields, whose conformal dimensions are 0 and $\frac{1}{2}$.

According to [20, 21, 26], the corresponding partition function of the transverse Ising model with an $x$-direction boundary field changes as a function of the magnitude of the boundary field. The $x$-direction and the $z$-direction boundary fields of the present model correspond to the magnetic field and the exchange interaction in the boundary row in the half-plane lattice, in terms of the corresponding classical system, namely the two-dimensional Ising model. This correspondence implies that there is a difference in the behaviour of the partition function. We discuss this point in more detail later.

We remark that the following relationship holds:

$$Z_{XY}(h=0) = \left(Z_{\text{2D Ising}}\right)^2 . \tag{6.3}$$

Namely, for any $h$, the transverse Ising model is equivalent to the two-dimensional Ising model ($c=\frac{1}{2}$), and for $h=0$ the $XY$ model is equivalent to the double two-dimensional Ising models ($c=\frac{1}{2}+\frac{1}{2}=1$). We discuss the operator contents in the $XY$ model with $h>0$, in the following two sections.

## 7. Review of the shifted $U(1)$ Kac–Moody algebra

In order to discuss the operator algebra of the isotropic $XY$ model with a boundary field, we need to recall the shifted $U(1)$ Kac–Moody algebra [28]. In the present section, we briefly review the algebra as a preliminary.

As is well known, the $U(1)$ Kac–Moody algebra is defined by the commutation relations

$$\begin{aligned} &[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0} \\ &[J_m, L_n] = mJ_{m+n} \qquad [J_m, J_n] = m\delta_{m+n,0} \end{aligned} \tag{7.1}$$

where $L_m^\dagger = L_{-m}$ and $J_m^\dagger = J_{-m}$ with $m, n \in \mathbb{Z}$. These generators of the present algebra, $\{L_m\}$ and $\{J_m\}$, are associated with the stress–energy tensor $T(z)$ and the current $J(z)$ as follows:

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2}L_n \qquad J(z) = \sum_{n\in\mathbb{Z}} z^{-n-1}J_n. \tag{7.2}$$

In other words

$$L_n = \oint_0 \frac{\mathrm{d}z}{2\pi\mathrm{i}} z^{n+1}T(z) \qquad J_n = \oint_0 \frac{\mathrm{d}z}{2\pi\mathrm{i}} z^n J(z). \tag{7.3}$$

We remark that the following relationship (the so-called Sugawara construction) holds:

$$L_m = \frac{1}{2}\sum_{n\in\mathbb{Z}} : J_{m-n}J_n : . \tag{7.4}$$

Here, we introduce the $U(1)$-charge operator $p$,

$$p \equiv \oint_0 \frac{\mathrm{d}z}{2\pi\mathrm{i}} J(z) = J_0. \tag{7.5}$$

In fact, the relation

$$\left[T(z), p\right] = 0 \tag{7.6}$$

means that $p$ describes the conserved charge. Next, we introduce an operator $q$, which is the canonical conjugate operator of $p$, namely

$$\left[q, p\right] = \mathrm{i}. \tag{7.7}$$

By using the operator $q$, we define the following operators:

$$L_m(\theta) \equiv e^{-i\theta q} L_m e^{i\theta q} \qquad \text{and} \qquad J_m(\theta) \equiv e^{-i\theta q} J_m e^{i\theta q}. \tag{7.8}$$

We can derive the explicit form of these operators by using the above commutation relations and the Sugawara construction formula, as follows:

$$L_m(\theta) = L_m + \theta J_m + \tfrac{1}{2}\theta^2 \delta_{m,0} \qquad \text{and} \qquad J_m(\theta) = J_m + \theta \delta_{m,0}. \tag{7.9}$$

The operators thus obtained fulfill the following relationships:

$$\begin{aligned} &[L_m(\theta), L_n(\theta)] = (m-n)L_{m+n}(\theta) + \frac{c}{12} m(m^2-1)\delta_{m+n,0} \\ &[J_m(\theta), L_n(\theta)] = mJ_{m+n}(\theta) \qquad [J_m(\theta), J_n(\theta)] = m\delta_{m+n,0} \end{aligned} \tag{7.10}$$

which are the same as the commutation relations of the $U(1)$ Kac–Moody algebra. The algebra generated by the operators $\{L_m(\theta)\}$ and $\{J_m(\theta)\}$ is called the shifted $U(1)$ Kac–Moody algebra [28].

We consider the following primary conformal field [29]:

$$U_\theta(z) = \exp(zL_{-1}) \exp(i\theta q) \exp(-zL_{-1}). \tag{7.11}$$

We evaluate the conformal dimensionality of the present operator. By definition, for $z \to w$, we have

$$T(z)U_\theta(w) = \frac{\Delta_\theta}{(z-w)^2} U_\theta(w) + \frac{1}{(z-w)} \partial_w U_\theta(w) + \cdots \tag{7.12}$$

where $\Delta_\theta$ denotes the conformal dimensionality of $U_\theta(z)$. Therefore, we find the relationship

$$\langle 0|U_\theta^\dagger(0)T(z)U_\theta(0)|0\rangle = \frac{\Delta_\theta}{z^2}. \tag{7.13}$$

By evaluating the left-hand side of this formula, we can obtain $\Delta_\theta$. In fact, since the following relation holds:

$$U_\theta^\dagger(0)T(z)U_\theta(0) = \sum_{n\in\mathbb{Z}} z^{-n-2}(L_n + \theta J_n) + z^{-2}\tfrac{1}{2}\theta^2 \tag{7.14}$$

we have

$$\langle 0|U_\theta^\dagger(0)T(z)U_\theta(0)|0\rangle = \frac{\theta^2/2}{z^2}. \tag{7.15}$$

Thus we obtain

$$\Delta_\theta = \tfrac{1}{2}\theta^2. \tag{7.16}$$

We define the highest-weight state $|\Delta, \phi\rangle$, by

$$\begin{aligned} &J_0|\Delta,\phi\rangle = \phi|\Delta,\phi\rangle \qquad L_0|\Delta,\phi\rangle = \Delta|\Delta,\phi\rangle \\ &J_m|\Delta,\phi\rangle = L_m|\Delta,\phi\rangle = 0 \qquad (m \geqslant 1). \end{aligned} \tag{7.17}$$

Then, from the Sugawara construction formula, we find

$$\Delta = \tfrac{1}{2}\phi^2. \tag{7.18}$$

We can evaluate the character function [28] in the shifted $U(1)$ Kac–Moody algebra with $c = 1$, which corresponds to the irreducible representation specified by $\phi$

$$\chi_\phi(y,z) \equiv \mathrm{tr}\left(y^{L_0(\theta)-\frac{c}{24}} z^{J_0(\theta)}\right) = \frac{1}{\eta(y)} y^{\frac{1}{2}(\phi+\theta)^2} z^{\phi+\theta} \tag{7.19}$$

where

$$L_0(\theta) = L_0 + \theta J_0 + \tfrac{1}{2}\theta^2 \qquad J_0(\theta) = J_0 + \theta \tag{7.20}$$

$$\eta(y) \equiv y^{\frac{1}{24}} \prod_{n=1}^{\infty}(1 - y^n). \tag{7.21}$$

Here, the symbol 'tr' means the summation over all the secondary fields generated from the primary field specified by $\phi$. From this character function (7.19), the conformal dimensionality $\Delta(\phi;\theta)$ of the primary field is concluded to be given by

$$\Delta(\phi;\theta) = \tfrac{1}{2}(\phi + \theta)^2. \tag{7.22}$$

Every quantum system on a chain with boundaries, which gives a representation of the algebra, is described by the Hamiltonian

$$\hat{H}(\theta) = \frac{\pi}{L} L_0(\theta) - \frac{\pi c}{24L} \tag{7.23}$$

where we subtract the non-universal bulk and the surface terms. Consequently, we can evaluate the finite-size correction for the ground-state energy as follows:

$$\langle 0|\hat{H}(\theta)|0\rangle = \frac{\pi}{L}\frac{\theta^2}{2} - \frac{\pi c}{24L} \tag{7.24}$$

where we have used $L_0|0\rangle = 0$ and $J_0|0\rangle = 0$ for the vacuum $|0\rangle \equiv |\Delta = 0, \phi = 0\rangle$. For $\theta = 0$, the present result reduces to the well known relationship derived in [14, 15]. We find that the term $\frac{\pi}{L}\frac{\theta^2}{2}$ corresponds to the conformal dimensionality of the ground state. In fact, according to the character function, the lowest conformal dimension equals to $\Delta(\phi = 0;\theta) = \frac{1}{2}\theta^2$. This dimensionality comes from the primary field $U_\theta(z)$. In the next section, we identify the parameter $\theta$ as the effect of the boundary field.

## 8. Operator contents of the $XY$ model

In the present section, we discuss the operator algebra of the isotropic $XY$ model with a boundary field.

At first we identify the parameter $\theta$ in the shifted $U(1)$ Kac–Moody algebra [28] as the function $\Theta(2h)/2$ in the isotropic $XY$ model. As is well known [10], the chiral Dirac field gives a representation of the $U(1)$ Kac–Moody algebra with $c = 1$. By using the generators of this algebra represented by the Dirac field, we can describe $\hat{H}(\theta)$. By comparing the Hamiltonian $\hat{H}(\theta)$ thus obtained and the low-lying effective Hamiltonian $\mathcal{H}_{\text{eff}}$ of the isotropic $XY$ model, we can directly identify the parameter $\theta$ as $\Theta(2h)/2$ in the following way.

We consider the chiral Dirac field

$$\Psi(z) = \sum_{n\in\mathbb{Z}} \psi_n z^{-n-\frac{1}{2}} \tag{8.1}$$

where

$$\psi_n = \frac{1}{\sqrt{2}}\left(\psi_n^{(1)} + i\psi_n^{(2)}\right) \qquad \left\{\psi_m^{(i)}, \psi_n^{(j)}\right\} = \delta_{m+n,0}\delta_{i,j}. \tag{8.2}$$

Here, $\psi_n^{(i)\dagger} = \psi_{-n}^{(i)}$. Since the stress–energy tensor and the current are described as

$$T(z) = -:\Psi^\dagger(z)\partial\Psi(z): \qquad J(z) =: \Psi^\dagger(z)\Psi(z): \tag{8.3}$$

we can evaluate the generators of the $U(1)$ Kac–Moody algebra, namely $\{L_m\}$, $\{J_m\}$, following their definitions. In particular, we have

$$L_0 = \sum_{m\in\mathbb{Z}} \left(m+\tfrac{1}{2}\right) : \psi^\dagger_{-m}\psi_m : \qquad J_0 = \sum_{m\in\mathbb{Z}} : \psi^\dagger_{-m}\psi_m : . \tag{8.4}$$

Here, we introduce new fermion operators as follows:

$$\xi_m = \psi_m,\ \xi^\dagger_m = \psi^\dagger_{-m} \qquad \text{and} \qquad \eta^\dagger_m = \psi_{-m-1},\ \eta_m = \psi^\dagger_{-m-1} \qquad \text{for } m = 0, 1, \ldots . \tag{8.5}$$

Then the operators $\{\xi_m\}$ and $\{\eta_m\}$ satisfy the fermionic anticommutation relation. By using these operators, a representation of the Hamiltonian $\hat{H}(\theta)$ (7.23) in terms of the Dirac field is given as follows:

$$\hat{H}_{\rm D}(\theta) = \frac{\pi}{L}\left\{\sum_{m=0}^{\infty}\left(m+\tfrac{1}{2}+\theta\right)\xi^\dagger_m\xi_m + \sum_{m=0}^{\infty}\left(m+\tfrac{1}{2}-\theta\right)\eta^\dagger_m\eta_m\right\} - \frac{\pi}{24L}\left(1-12\theta^2\right). \tag{8.6}$$

This takes the same form as the low-lying effective Hamiltonian of the isotropic $XY$ model (3.11), when $\theta = \Theta(2h)/2$. Namely, we can identify the effect of the boundary field as the parameter $\theta$ in the $U(1)$ Kac–Moody algebra. In other words, we find that the present isotropic $XY$ model is also a representation of the $U(1)$ Kac–Moody algebra. In fact, by tracing the above calculation inversely, we can describe all the generators of the algebra in terms of the fermions in the isotropic $XY$ model (3.11).

Moreover, we evaluate the partition function of the Dirac field as follows,

$$\begin{aligned} Z_{\rm D} &= q^{-\frac{1}{24}} q^{\frac{\theta^2}{2}} \prod_{m=0}^{\infty}\left(1+q^{m+\frac{1}{2}+\theta}\right)\prod_{m=0}^{\infty}\left(1+q^{m+\frac{1}{2}-\theta}\right) \\ &= \frac{1}{\eta(q)}\sum_{n\in\mathbb{Z}} q^{\frac{1}{2}(n+\theta)^2} \end{aligned} \tag{8.7}$$

where $q \equiv \mathrm{e}^{-\pi/LT}$. The last form of $Z_{\rm D}$ is described as

$$Z_{\rm D} = \sum_{n\in\mathbb{Z}} \chi_n(q, 1) \tag{8.8}$$

with $\chi_n(q, 1)$ defined by (7.19). Therefore we obtain the conformal weight of each primary field in the present Dirac field as follows:

$$\Delta(n;\theta) = \tfrac{1}{2}(n+\theta)^2 \qquad n \in \mathbb{Z}. \tag{8.9}$$

Next, we construct $\hat{H}(\theta)$ in terms of the chiral Gaussian field and evaluate the partition function, so that we compare the partition function thus obtained with $Z_{XY}$. As is well known [10], the present field also gives a representation of the $U(1)$ Kac–Moody algebra with $c = 1$. The chiral Gaussian field takes the following form [27], which is compacted to a circle of radius $R$:

$$\Phi(z) = q - \mathrm{i}p\ln z + \mathrm{i}\sum_{n\neq 0}\frac{\alpha_n}{n}z^{-n} \tag{8.10}$$

where

$$[q, p] = \mathrm{i} \qquad [\alpha_m, \alpha_n] = m\delta_{m+n,0}. \tag{8.11}$$

Then, the allowed eigenvalues of $p$ are given by $M/R$ ($M \in \mathbb{Z}$). The stress–energy tensor and the current are described as

$$T(z) = -\tfrac{1}{2} : (\partial\Phi(z))^2 : \qquad J(z) = \mathrm{i}\partial\Phi(z). \tag{8.12}$$

Thus we can represent the generators of the $U(1)$ Kac–Moody algebra in terms of the present field. In particular we have

$$L_0 = \tfrac{1}{2}\alpha_0^2 + \sum_{n=1}^{\infty} \alpha_{-n}\alpha_n \qquad J_0 = \alpha_0 \tag{8.13}$$

where $\alpha_0 \equiv p$. Here, we introduce boson operators as follows:

$$a_n = \frac{1}{\sqrt{n}}\alpha_n \qquad a_n^\dagger = \frac{1}{\sqrt{n}}\alpha_{-n} \qquad \text{for } n = 1, 2, \ldots. \tag{8.14}$$

Then the operators $\{a_n\}$ satisfy the bosonic commutation relation. By using these operators, a representation of the Hamiltonian $\hat{H}(\theta)$ (7.23) in terms of the Gaussian field is given as follows:

$$\hat{H}_{\mathrm{G}}(\theta) = \frac{\pi}{L}\left\{\sum_{n=1}^{\infty} n a_n^\dagger a_n + \frac{1}{2}\left(\frac{M}{R} + \theta\right)^2\right\} - \frac{\pi}{24L}. \tag{8.15}$$

We can evaluate the partition function of the Gaussian field as follows:

$$\begin{aligned} Z_{\mathrm{G}}(R) &= q^{-\frac{1}{24}} \prod_{n=1}^{\infty} \frac{1}{(1-q^n)} \sum_{M\in\mathbb{Z}} q^{\frac{1}{2}(\frac{M}{R}+\theta)^2} \\ &= \frac{1}{\eta(q)} \sum_{M\in\mathbb{Z}} q^{\frac{1}{2}(\frac{M}{R}+\theta)^2} \end{aligned} \tag{8.16}$$

where $q \equiv \mathrm{e}^{-\pi/LT}$. The last form of $Z_{\mathrm{G}}$ is described as

$$Z_{\mathrm{G}}(R) = \sum_{M\in\mathbb{Z}} \chi_{M/R}(q, 1). \tag{8.17}$$

with $\chi_{M/R}(q, 1)$ defined by (7.19). Therefore we obtain the conformal weight of each primary field in the Gaussian field as follows:

$$\Delta(M/R; \theta) = \frac{1}{2}\left(\frac{M}{R} + \theta\right)^2 \qquad M \in \mathbb{Z}. \tag{8.18}$$

We find the following relationship:

$$Z_{\mathrm{G}}(R = 1) = Z_{\mathrm{D}} \qquad \text{for } \forall\, \theta. \tag{8.19}$$

This relationship is extremely natural since the both fields are directly connected with each other by the bosonization formula [11, 12]

$$\Psi(z) =: \exp \mathrm{i}\Phi(z) : . \tag{8.20}$$

Then we find that the following relationship holds:

$$Z_{\mathrm{G}}(R = 1) = Z_{XY} \tag{8.21}$$

when we identify $\theta$ as $\Theta(2h)/2$.

Through the discussions in the present section, we find that the isotropic $XY$ model with a boundary field gives a representation of the shifted $U(1)$ Kac–Moody algebra, which is a Virasoro algebra with $c = 1$. Namely, $Z_{XY}$ is described by the character functions of the algebra as follows

$$Z_{XY} = \sum_{n \in \mathbb{Z}} \chi_n(q, 1) \tag{8.22}$$

where $\theta = \Theta(2h)/2$. Therefore we obtain the conformal dimensions of all the primary fields to be

$$\Delta(n; \Theta(2h)/2) = \frac{1}{2}\left(n + \frac{\Theta(2h)}{2}\right)^2. \tag{8.23}$$

The dimensionality changes as a function of the boundary field.

## 9. Continuum limit of the models

In the present section, we consider the continuum limit of the two models, namely the transverse Ising model and the $XY$ model, with boundary fields. Then we explain qualitatively why the conformal dimensions change (do not change) in the former (latter) model.

As is well known [10], the Lagrangian density of the critical transverse Ising model in the continuum limit is described as the massless Majorana fermion fields $\{\psi, \bar{\psi}\}$ as follows:

$$\mathcal{L}_{\mathrm{M}} = \bar{\psi}\,(\partial_\tau - \mathrm{i}\partial_x)\,\bar{\psi} + \psi\,(\partial_\tau + \mathrm{i}\partial_x)\,\psi. \tag{9.1}$$

Here, we take the Euclidean coordinate $(\tau, x)$. The relevant model is defined on the strip $\Omega$ with the width $L$, namely $0 \leqslant x \leqslant L$ and $-\infty < \tau < +\infty$. Then, in terms of the Majorana field, the boundary field $\mathcal{B}$ is described as follows:

$$\mathcal{B} = h\sigma_1^z + h\sigma_L^z \sim h \sum_{\partial\Omega} \mathrm{i}\bar{\psi}\psi. \tag{9.2}$$

Here, the symbol $\partial\Omega$ denotes the edges of the present strip, namely

$$\{(\tau, x) | \tau \in (-\infty, +\infty), x = 0 \text{ or } L\}.$$

Therefore, the sum means the integration on the edges.

In contrast, as is also well known [10], the Lagrangian density of the isotropic $XY$ model in the continuum limit is described as the massless Dirac fermion field. Moreover, by bosonization [11, 12], we can obtain the following bosonic form, namely a massless scalar field $\varphi$:

$$\mathcal{L}_{\mathrm{S}} = \tfrac{1}{2}\left(\partial_\mu \varphi\right)^2 \qquad \partial^2 = \partial_t^2 - \partial_x^2. \tag{9.3}$$

In this model, we take the Minkowskian coordinate $(t, x)$. The relevant model is also defined on the strip $\Omega$ with the width $L$, namely $0 \leqslant x \leqslant L$ and $-\infty < t < +\infty$. Then, in terms of this scalar field, the boundary field $\mathcal{B}$ is described as follows:

$$\mathcal{B} \sim h \sum_{\partial\Omega} \partial_x \varphi - h \sum_{\partial\Omega_1} M : \cos\sqrt{4\pi}\varphi : + h \sum_{\partial\Omega_2} M : \cos\sqrt{4\pi}\varphi : \tag{9.4}$$

where $M$ is an arbitrary constant with the mass dimensions. The symbol $\partial\Omega_1$ and $\partial\Omega_2$ denote the both edges of the strip $\Omega$, namely they correspond to $\{(t, x) | t \in (-\infty, +\infty), x = 0\}$ and $\{(t, x) | t \in (-\infty, +\infty), x = L\}$, respectively. Therefore, formally we have $\partial\Omega = \partial\Omega_1 + \partial\Omega_2$.

The above two continuum models are defined on the strip $\Omega$, which is described as $\{(x_0, x_1)|x_0 \in (-\infty, +\infty), x_1 \in [0, L]\}$, where $x_0 = \tau$ or $t$ and $x_1 = x$. By the transformation

$$z = \exp\left\{\frac{\pi}{L}(x_0 + \mathrm{i}x_1)\right\} \tag{9.5}$$

we deform the strip into one half of the complex plane of $z$ with $\operatorname{Im} z \geqslant 0$. Then, especially $\partial\Omega_1$ and $\partial\Omega_2$ are transformed into the real axis with $\operatorname{Re} z > 0$ and that with $\operatorname{Re} z < 0$, respectively.

In the present Majorana fermion field on the half $z$ plane, we find that the boundary field $\mathrm{i}\bar{\psi}\psi$ does not violate the $Z_2$ symmetry, because this boundary field corresponds to the energy-density operator in terms of the two-dimensional Ising model. That may be one of the reasons why the conformal dimensions of the primary fields, which the present model contains, do not change.

On the other hand, in the present scalar field on the half $z$ plane, the boundary field $\partial_x\varphi$ does not violate the $U(1)$ symmetry. However, the field proportional to $: \cos\sqrt{4\pi}\varphi :$ violates the symmetry. Moreover, there exists a discontinuity of the boundary field at $z = 0$, unless $\varphi$ takes specific values. We expect that this is one of the reasons why the conformal dimensions of the primary fields, which the present model contains, vary.

## 10. Summary

The critical transverse Ising model on an open chain gives a representation of the Virasoro algebra with $c = \frac{1}{2}$ for any $h$. That is, the conformal dimensions of the primary fields do not change, even if the boundary field becomes finite. In contrast, the isotropic $XY$ model with a boundary field gives a representation of the $U(1)$ Kac–Moody algebra which is a Virasoro algebra with $c = 1$. The conformal dimensions of the primary fields change as a function of the boundary field. The difference between these models may come from the difference of the property of the boundary field emerging on the edge of the half-plane after the transformation. In terms of the original lattice model, the boundary field at the end points takes the same form in both models. However, since each of these models has different symmetry, the role of the boundary field is also different.

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## Appendix A

Here we explain how to diagonalize the $L \times L$ matrix $M$ shown in section 2.

The present matrix of the Hamiltonian (2.2) takes the following form:

$$M = \begin{bmatrix} b_0 & b_1 & a_2 & 0 & 0 & & & \mathbf{0} \\ b_1 & a_0 & a_1 & a_2 & 0 & & & \\ a_2 & a_1 & a_0 & a_1 & a_2 & & & \\ \cdots & & \cdots & & \cdots & & \cdots & \\ & & & & a_2 & a_1 & a_0 & c_1 \\ \mathbf{0} & & & & 0 & a_2 & c_1 & c_0 \end{bmatrix} \tag{A.1}$$

where

$$\begin{aligned} a_0 &= 4\gamma^2 + 2(1+\alpha^2) \qquad & a_1 &= 4\gamma \qquad a_2 = 1-\alpha^2 \\ b_0 &= 4(\gamma+h)^2 + (1-\alpha)^2 \qquad & b_1 &= 2h(1+\alpha) + 4\gamma \\ c_0 &= 4(\gamma+h')^2 + (1+\alpha)^2 \qquad & c_1 &= 2h'(1-\alpha) + 4\gamma. \end{aligned} \tag{A.2}$$

We consider here only the following two cases, namely $\alpha = \gamma = 1$ (the critical transverse Ising model) and $\alpha = \gamma = 0$ (the isotropic $XY$ model).

In the case of $\alpha = \gamma = 1$, we assume that the $n$th component of the eigenfunction takes the following form:

$$f_n(k) = A_+(k)\mathrm{e}^{\mathrm{i}k(L-n+1)} + A_-(k)\mathrm{e}^{-\mathrm{i}k(L-n+1)} \qquad n = 2, 3, \ldots, L-1. \tag{A.3}$$

Then, with respect to higher-order from the 3rd up to the $(L-2)$nd components, the eigenvalue equation $Mf(k) = E(k)f(k)$ holds, where $E(k) = (4\cos(k/2))^2$. Moreover, when the relationship

$$\begin{pmatrix} a & \bar{a} \\ b & \bar{b} \end{pmatrix} \begin{pmatrix} A_+ \\ A_- \end{pmatrix} = 0 \tag{A.4}$$

$$\begin{aligned} a &= \left(1 + \mathrm{e}^{-\mathrm{i}k} - (1+h')^2\right)\mathrm{e}^{\mathrm{i}k} \\ b &= \left(2 + \mathrm{e}^{\mathrm{i}k} + \mathrm{e}^{-\mathrm{i}k} - (1+h)^2(1+\mathrm{e}^{-\mathrm{i}k})\right)\mathrm{e}^{\mathrm{i}kL} \end{aligned} \tag{A.5}$$

is satisfied, the eigenvalue equation holds for all the components. Then, $f_1$ and $f_L$ are expressed by using $A_+$, $A_-$ and $k$. In order that a non-trivial solution $(A_+, A_-) \neq (0, 0)$ exists, the determinant of this $2 \times 2$ matrix in (A.4) must be zero. This condition leads us to (2.6), the equation that determines the allowed $k$.

In the case of $\alpha = \gamma = 0$, we assume that the $n$th component of the eigenfunction takes the following form:

$$f_n(k) = \begin{cases} A_+(k)\mathrm{e}^{\mathrm{i}kn} + A_-(k)\mathrm{e}^{-\mathrm{i}kn} & n = 4, 6, \ldots L-2 \\ B_+(k)\mathrm{e}^{\mathrm{i}k(L-n+1)} + B_-(k)\mathrm{e}^{-\mathrm{i}k(L-n+1)} & n = 3, 5, \ldots, L-3. \end{cases} \tag{A.6}$$

In this case, in order that the eigenvalue equation $Mf(k) = E(k)f(k)$, where $E(k) = (2\cos(k))^2$, should hold for all the components, the following relation must be satisfied:

$$\begin{pmatrix} 1 & 1 & -b & -\bar{b} \\ c & \bar{c} & -a & -\bar{a} \\ -b' & -\bar{b}' & 1 & 1 \\ -a' & -\bar{a}' & c' & \bar{c}' \end{pmatrix} \begin{pmatrix} A_+ \\ A_- \\ B_+ \\ B_- \end{pmatrix} = 0 \tag{A.7}$$

$$\begin{aligned} a &= (1 + \mathrm{e}^{2\mathrm{i}k} - 4h^2)\mathrm{e}^{\mathrm{i}kL} \qquad & b &= 2h\mathrm{e}^{\mathrm{i}kL} \qquad & c &= 2h\mathrm{e}^{2\mathrm{i}k} \\ a' &= (1 + \mathrm{e}^{2\mathrm{i}k} - 4h'^2)\mathrm{e}^{\mathrm{i}kL} \qquad & b' &= 2h'\mathrm{e}^{\mathrm{i}kL} \qquad & c' &= 2h'\mathrm{e}^{2\mathrm{i}k}. \end{aligned} \tag{A.8}$$

The condition that the determinant of this $4 \times 4$ matrix in (A.7) should be zero gives us the equation of the allowed $k$ (2.8).

## Appendix B

Here we briefly discuss the solutions of the equations of $k$, namely (2.6) and (2.8) with $h = h'$.

At first, we consider the critical transverse Ising model. We can find $L$ real solutions of eq. (2.6) in the region $0 < k \leqslant \pi$, for $0 \leqslant \varepsilon < 1$, where $\varepsilon \equiv (1+h)^2 - 1$. In fact, by using the following rewritten form of (2.6):

$$\frac{1}{L}\frac{1}{\pi \mathrm{i}} \ln \frac{1-\varepsilon \mathrm{e}^{-\mathrm{i}k_m}}{1-\varepsilon \mathrm{e}^{+\mathrm{i}k_m}} = \frac{m}{L} - \frac{k_m}{\pi}\left(1+\frac{1}{2L}\right) \qquad m = 1, 2, \ldots, L \tag{B.1}$$

we can confirm that $L$ solutions exist on the real axis of the complex $k$ plane with $0 < k \leqslant \pi$. However, for $\varepsilon > 1$, we obtain only $L-2$ real solutions in the region $0 < k \leqslant \pi$. The remaining two solutions exist on the imaginary axis of the complex $k$ plane, and takes the following form:

$$k = \pm \mathrm{i}\kappa \qquad \mathrm{e}^{\kappa} = \varepsilon + \delta_L. \tag{B.2}$$

Here, the symbol $\delta_L$ denotes the correction term which depends on $L$, and is proportional to $\varepsilon^{-L}$. Therefore, we find that the finite-size corrections of $\mathrm{e}^{\pm\kappa}$, with respect to the the powers of $1/L$, are equal to zero.

Next, we discuss the isotropic $XY$ model. In the following rewritten form of (2.8):

$$\frac{1}{L}\frac{1}{\pi \mathrm{i}} \ln \frac{1 \pm \varepsilon \mathrm{e}^{-\mathrm{i}k_m}}{1 \pm \varepsilon \mathrm{e}^{+\mathrm{i}k_m}} = \frac{m}{L} - \frac{k_m}{\pi}\left(1+\frac{1}{L}\right) \qquad m = 1, 2, \ldots, \frac{L}{2}. \tag{B.3}$$

For $0 \leqslant \varepsilon < 1$ with $\varepsilon \equiv 2h$, we find $L$ real solutions in the region $0 < k \leqslant \pi$, as for the previous model. For $\varepsilon > 1$, the present model also has $L-2$ real solutions and two pure imaginary solutions. The imaginary solutions take the same form, equation (B.2), as those of the previous case, though the definition of $\varepsilon$ is different.

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